

A categorical framework for cellular automata

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Abstract. This paper proposes a generalized framework for cellular automata using the language of category theory, extending the classical definition beyond set-theoretic constraints. For an arbitrary category $\mathcal{C}$ with products, we define $\mathcal{C}$ -cellular automata as morphisms $\tau : A^G \rightarrow B^G$ in $\mathcal{C}$, where the alphabets A and B are objects in $\mathcal{C}$ and the universe is a group G . We show that $\mathcal{C}$ -cellular automata form a subcategory of $\mathcal{C}$ closed under finite products, and that they satisfy a categorical version of the Curtis-Hedlund-Lyndon theorem. For two arbitrary group universes G and H , we extend our theory to define generalized $\mathcal{C}$ -cellular automata as morphisms $\tau : A^G \rightarrow B^H$ in $\mathcal{C}$ constructed via a group homomorphism $\phi : H \rightarrow G$. Finally, we prove that generalized $\mathcal{C}$ -cellular automata form a subcategory of $\mathcal{C}$ with a finite weak product involving the free product of the underlying group universes. This framework unifies existing concepts and provides purely categorical proofs of foundational results in the theory of cellular automata.

Keywords: Cellular automata · Generalized cellular automata · Category theory · Categorical product · Curtis-Hedlund-Lyndon theorem.

1 Introduction

The central categorical concept of this work is the universal property of the product. A category $\mathcal{C}$ consists of a collection of objects, a collection of morphisms, and an associative composition rule with an identity morphism for each object in the category. There are many categories in mathematics, such as **Set**, where objects are sets, morphisms are functions, and the composition is the usual composition of functions; **Grp** for groups, group homomorphisms and the usual composition; **Top** for topological spaces and continuous functions; and so on.

Given a category $\mathcal{C}$ and a family $\{A_i\}_{i \in I}$ of objects of $\mathcal{C}$, we say that the family has a product if there exists an object P of $\mathcal{C}$ and a family $\{P \xrightarrow{\pi_i} A_i\}_{i \in I}$ of morphisms of $\mathcal{C}$ (called projections) such that for every object Q and every family $\{Q \xrightarrow{f_i} A_i\}_{i \in I}$ of morphisms there exists a unique morphism $f : Q \rightarrow P$ such that $\pi_i \circ f = f_i$ for all $i \in I$. This existence and uniqueness condition is known as the universal property of the product. We now explain how the critical constructions needed to discuss cellular automata in the category $\mathcal{C}$ arise as applications of this universal property.

2 $\mathcal{C}$ -cellular automata

Consider a category $\mathcal{C}$ in which all families of objects have products. Thus, for an object A of $\mathcal{C}$, and a set I , we consider the product A^I of the I -indexed family whose objects are all copies of A and whose projections are denoted by $\pi_i^I : A^I \rightarrow A$, for every $i \in I$. The first construction necessary for cellular automata is the restriction morphism. For any subset $S \subseteq I$, the universal property of the product A^S gives us a unique morphism $\text{Res}_S^I : A^I \rightarrow A^S$ such that $\pi_s^I = \pi_s^S \circ \text{Res}_S^I$.

Another important construction is the shift action. In general, given a function $f : I \rightarrow J$ of sets, fixing an object A in $\mathcal{C}$, by the universal property of the product A^I , there exists a unique morphism $f_A^* : A^J \rightarrow A^I$ such that $\pi_{f(i)}^I = \pi_i^J \circ f_A^*$. In particular, given a group G , we have for each $g \in G$ the right translation $R_g : G \rightarrow G$ defined by $R_g(x) := xg$ for all $x \in G$, and so we have the morphism $R_g^* : A^G \rightarrow A^G$. This gives us a group homomorphism $\varphi : G \rightarrow \text{Aut}(A^G)$ and we will denote the assignment of g as φ_g that represents the translation by g in the shift action. This yields the following definition.

Definition 1 ($\mathcal{C}$ -cellular automaton). *Let A^G and B^G be two objects in $\mathcal{C}$, and let $\tau : A^G \rightarrow B^G$ be a morphism. We say that τ is a $\mathcal{C}$ -cellular automaton over G if there exists a finite subset $S \subseteq G$ and a morphism $\mu : A^S \rightarrow B$ in $\mathcal{C}$ such that*

$$\pi_g \circ \tau = \mu \circ \text{Res}_S^G \circ \varphi_g \quad \text{for all } g \in G.$$

3 Categorical Curtis-Hedlund-Lyndon theorem

One of the most important results of the paper is a categorical version of the classical Curtis-Hedlund-Lyndon theorem. We introduce the notion of a *uniform morphism*. A morphism $f : A^I \rightarrow B^I$ is uniform if for every $i \in I$, the morphism $\pi_i \circ f : A^I \rightarrow B$ depends only on finitely many coordinates. This notion generalizes the classical idea of continuity in the prodiscrete topology.

The main theorem states that a morphism $\tau : A^G \rightarrow B^G$ is a $\mathcal{C}$ -cellular automaton if and only if it is G -equivariant (This means that it commutes with φ_g for every $g \in G$) and uniform. This result shows that the categorical definition captures exactly the same essential behavior as classical cellular automata do, but without requiring topological or set-theoretic assumptions.

4 Products of $\mathcal{C}$ -cellular automata

We also study the categorical structure of cellular automata. We prove that the collection of configuration objects and $\mathcal{C}$ -cellular automata form a subcategory of $\mathcal{C}$, denoted by $\mathbf{CA}_{\mathcal{C}}(G)$. Moreover, this category is closed under finite products.

Given two configuration objects A^G and B^G , their product in $\mathbf{CA}_{\mathcal{C}}(G)$ is given by $(A \times B)^G$ where the projections are obtained by a construction (using the universal property of the product) from the projections of the product $A \times B$ in $\mathcal{C}$.

5 Generalized $\mathcal{C}$ -cellular automata and their weak products

Finally, we extend the theory to generalized cellular automata between different universes. Given groups G and H , a generalized cellular automaton is a morphism $\tau : A^G \rightarrow B^H$ associated with a group homomorphism $\phi : H \rightarrow G$.

We prove that generalized $\mathcal{C}$ -cellular automata form a category $\mathbf{GCA}_{\mathcal{C}}$. Furthermore, we show that this category admits weak products; that is, products satisfying the universal property without requiring uniqueness of the induced morphism. The weak product of A^G and B^H is given by $(A \times B)^{G * H}$, where $G * H$ denotes the free product of groups. The projections are obtained by combining the injections into the free product with the projections of the product $A \times B$ in $\mathcal{C}$.

These results provide a unified and highly abstract framework for studying cellular automata, extending many classical theorems into the setting of category theory.

References

1. Castillo-Ramirez, A., Sanchez-Alvarez, M., Vazquez-Acevez, A., Zaldivar-Corichi, A.: A generalization of cellular automata over groups. *Communications in Algebra* **51**(7), 3114–3123 (2023).
2. Castillo-Ramirez, A., Vazquez-Aceves, A., Zaldivar-Corichi, A.: Categorical Products of Cellular Automata. In: Riva, S., Richard, A. (eds) *Cellular Automata and Discrete Complex Systems. AUTOMATA 2025. Lecture Notes in Computer Science* **15831**. Springer, Cham. (2026).
3. Castillo-Ramirez, A., Vazquez-Aceves, A., Zaldivar-Corichi, A.: A categorical framework for cellular automata. <https://doi.org/10.48550/arXiv.2602.04049> (2026).